THE UNIVERSITY OF ALBERTA

SOME PROBLEMS ON THE STRUCTURE
OF ROUND-ROBIN TOURNAMENTS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES & RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

SPRING, 1972

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES & RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies & Research for acceptance, a thesis entitled SOME PROBLEMS ON THE STRUCTURE OF ROUND-ROBIN TOURNAMENTS submitted by MYRON GOLDBERG, B.Sc. (Hons.), M.Sc., in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

This thesis deals with four problems in the theory of round-robin tournaments. Most of the results can be formulated in terms of various kinds of mappings between the nodes or edges of two tournaments.

In Chapter 1, we define a binary operation on tournaments called <u>composition</u>. We develop various algebraic properties of this operation, and use these results to determine when the composition of two tournaments is commutative.

Let ϕ denote a one-to-one correspondence between the edges of two irreducible tournaments R and S. In Chapter 2, we obtain conditions on ϕ under which R and S are either isomorphic or can be made so by reversing the orientation of every edge of R or of S.

In Chapter 3, we determine the automorphism group of what we call the <u>quadratic residue tournament</u>. We use this result to determine when certain permutations of a finite field F are automorphisms of F, when $|F| \equiv 3 \pmod 4$.

We define a <u>k-irreducible tournament</u> in Chapter 4, and we prove that every node in such a tournament is contained in at least k distinct cycles of each length h, where $3 \leq h \leq n$.

ACKNOWLEDGMENTS

I would like to express my appreciation to my supervisor, Professor J.W. Moon, for his guidance during the preparation of this thesis. Special thanks are due Dr. R.D. Bercov, not only for conversations concerning points of algebra which arose in Chapter 3, but also for his interest and advice through most of my student career.

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CHAPTER 1

The Composition of Two Tournaments

§1. Introduction

A (round-robin) tournament T is a nonempty set of nodes $1,2,\cdots,n$ such that each pair of distinct nodes i and j is joined by exactly one of the oriented edges $\overrightarrow{ij}$ or $\overrightarrow{ji}$. If $\overrightarrow{ij}$ is in T, then we say that i dominates j and write $i \to j$; more generally, if every node of a subtournament A dominates every node of a subtournament B we write $A \to B$. We frequently use the same symbol to denote the set of nodes of a tournament and the tournament itself; expressions such as $A \subseteq B$, $A \cup B$ or $A \cap B$, where A and B are tournaments, should be interpreted with this convention in mind. For a general reference on tournaments, see [17].

We let $|X|$ denote the number of nodes in the tournament X. Two tournaments A and B are isomorphic if and only if there exists a one-to-one mapping ϕ between their nodes such that $i \to j$ in A if and only if $\phi(i) \to \phi(j)$ in B; we then write $A = B$. In this chapter, the symbol T_n will denote the transitive tournament whose n nodes can be labelled so that $i \to j$ if and only if $i > j$ for $1 \leq j < i \leq n$.

If the tournaments R and S have r and s nodes, respectively, then the composition of R with S is the tournament $R \circ S$ with rs nodes (i,k), where $1 \leq i \leq r$ and $1 \leq k \leq s$, such that $(i,k) \to (j,\ell)$ if and only if $i \to j$ in R or $i = j$ and $k \to \ell$ in S. In other words, $R \circ S$ is obtained by replacing each node i of R by a copy S_i of S and letting $S_i \to S_j$ in $R \circ S$ if and only if $i \to j$ in R.

The composition of two tournaments is associative but it is not commutative in general. For example, if R and S denote, respectively, the transitive tournament T_3 and the tournament with 3 nodes i,j,k where $i \to j$, $j \to k$ and $k \to i$, then $R \circ S$ contains a node which dominates seven other nodes but $S \circ R$ does not. Our main object here is to characterize those pairs of tournaments R and S for which $R \circ S = S \circ R$. We first develop various algebraic properties of the composition operations and another operation introduced in the next section. Lovász [16], Hemminger [12], Sabidussi [23], Imrich [14] and others have investigated algebraic properties of similar operations on other classes of graphs.

§2. Sums of Tournaments

If A and B are two subtournaments such that $A \to B$ then their sum is the tournament $A + B$ determined by the nodes of A and B. A tournament is reducible or irreducible according as it can or cannot be expressed as the sum of two smaller tournaments. A sequence of edges in a tournament of the type $\overrightarrow{ab}$, $\overrightarrow{bc}$, $\cdots$, $\overrightarrow{pq}$ determines a path $P(a,q)$ from a to q. We assume that the nodes $a,b,\cdots,q$ are all

different. If the edge $\overrightarrow{qa}$ is in the tournament, then the edges in $P(a,q)$ and the edge $\overrightarrow{qa}$ determine a cycle. The length of a path or cycle is the number of edges it contains. We regard a single node as a path of length zero or a cycle of length 1. A tournament is strongly connected if for every ordered pair of nodes p and q there exists a path from p to q. It is not difficult to prove (see [22]) that a tournament is irreducible if and only if it is strongly connected. Observe that the trivial tournament with one node is both irreducible and transitive.

Lemma 2.1. $A + (B+C) = (A+B) + C$.

Lemma 2.2. $(A+B)\circ C = (A\circ C) + (B\circ C)$.

Lemma 2.3. Any tournament R has a unique representation of the type $R = A + B + \cdots + K$, where the nonvacuous tournaments $A,B,\cdots,K$ are all irreducible.

The first two results are obvious; the third follows from repeated applications of the definition of irreducibility and from the definition of isomorphic. Note that composition is not left distributive with respect to addition.

Lemma 2.4. Suppose that $A + B = C + D$;

a) if $|A| = |C|$, then $A = C$ and $B = D$, and

b) if $|A| < |C|$, then $C = A + E$ and $B = E + D$ for some tournament E .

This follows immediately upon applying Lemma 2.3 to the tournament $R = A + B = C + D$.

Lemma 2.5. If $A + B = B + A$, then $A = T_k \circ C$ and $B = T_\ell \circ C$ for some tournaments T_k, T_ℓ , and C where T_k and T_ℓ are transitive.

Proof. If $|A| = |B|$, then $k = \ell = 1$ and $C = A = B$. If $|A| < |B| = m$ then by Lemma 2.4 (b), $B = A + E = E + A$ for some tournament E . It follows from the induction hypothesis that $A = T_i \circ C$ and $E = T_j \circ C$ for transitive tournaments T_i and T_j and some tournament C . But then $B = T_i \circ C + T_j \circ C = (T_i + T_j) \circ C = T_{i+j} \circ C$ and the lemma follows by induction.

Lemma 2.6. If $|A| > 1$, then $A \circ B$ is irreducible if and only if A is irreducible.

Proof. It is obvious that $A \circ B$ is strongly connected if and only if A is when $|A| > 1$. The result now follows from the fact that irreducibility is equivalent to strong connectedness in a tournament.

Lemma 2.7. If (a) $A + X = Y + B$ and

(b) $X + A + X = Y + B + Y$, then $X = Y = T_j \circ C$ and $A = B = T_k \circ C$ for some tournaments T_j, T_k and C where T_j and T_k are transitive.

Proof. Since $|X+A| = |Y+B|$ we have $X = Y$ and $A = B$ upon applying Lemma 2.4 to $(X+A) + X = (Y+B) + Y$ first and then to $X+A = Y+B$. The required result now follows from (a) and Lemma 2.5.

The following result will play an important role in the proof of one of our main theorems in §5.

Lemma 2.8. If (a) $A + X = Y + B$ and

(b) $X + [G \circ (A+X)] = [H \circ (Y+B)] + Y$, then $X = Y = T_j \circ C$, $A = B = T_k \circ C$ and $G = H = T_\ell$ for some tournaments T_j, T_k, T_ℓ and C where T_j, T_k and T_ℓ are transitive.

Proof. Since $|X| + |G| \cdot |A+X| = |Y| + |H| \cdot |A+X|$, by (a) and (b), it must be that $|X| = |Y|$, $|G| = |H|$, and $|A| = |B|$. If $|G| = |H| = 1$, the result follows from Lemma 2.7 so we suppose that $|G| = |H| > 1$.

Since $|X| < |H \circ (Y+B)|$, equation (b) and Lemma 2.4 imply that $X+W = H \circ (Y+B)$ for some tournament W ; that is, $H \circ (Y+B)$ is reducible. Therefore, H itself is reducible, by Lemma 2.6, and we may write $H = U+V$ where U is irreducible. Consequently, $X+W = (U+V) \circ (Y+B) = [U \circ (Y+B)] + [V \circ (Y+B)]$ by Lemma 2.2. Again, since $|X| < |U \circ (Y+B)|$, it must be that $X+S = U \circ (Y+B)$ for some tournament S . But $U \circ (Y+B)$ can be reducible only if $U = T_1$, by Lemma 2.6, since U is irreducible. Therefore, $H = T_1 + V$ and we also find, in a similar way, that $G = Z+T_1$ for some tournament Z . Hence,

$$X + [Z \circ (A+X)] + A + X = X + [G \circ (A+X)]$$

$$= [H \circ (Y+B)] + Y = Y + B + [V \circ (Y+B)] + Y .$$

Since $|X| = |Y|$, it follows from the preceding equation and Lemma 2.4 that $X = Y$ and $B + [V \circ (Y+B)] = [Z \circ (A+X)] + A$. Now $1 \leq |V| = |Z| < |G| = |H|$ so we may assume, as our induction hypothesis, that $X = Y = T_j \circ C$, $A = B = T_k \circ C$, and $V = Z = T_\ell$ for some tournaments T_j, T_k, T_ℓ and C , where T_j, T_k and T_ℓ are transitive. But

then $G = Z + T_1 = T_{\ell+1} = T_1 + V = H$, and the result now follows by induction.

§3. Preliminary Results

If $R \circ S = W \circ Z$, then there exists a one-to-one dominance-preserving mapping α of the nodes of $R \circ S$ onto the nodes of $W \circ Z$. In this section we derive some results on the nature of the image $\alpha(S_o)$ in $W \circ Z$ of any particular copy S_o of S in $R \circ S$, and of the preimage $\alpha^{-1}(Z_o)$ in $R \circ S$ of any particular copy Z_o of Z in $W \circ Z$.

Lemma 3.1. Let S_o and Z_o denote any particular copies of S and Z in $R \circ S$ and $W \circ Z$ where $R \circ S = W \circ Z$. If $S_o = A \cup X$ for disjoint tournaments A and X , and $Z_o = X' \cup B'$ for disjoint tournaments X' and B' , where $\alpha(X) = X'$ and $\alpha(A) \cap B' = \phi$, then either $A \to X$ and $X' \to B'$ or $X \to A$ and $B' \to X'$.

Proof. If some node $p \in A$ dominates some but not all of the nodes of X , then $\alpha(p)$ must have the same property with respect to $\alpha(X)$. But the only nodes in $W \circ Z$ that possibly have this property are in Z_o and $\alpha(p) \notin Z_o$. Therefore, either $p \to X$ or $X \to p$. Similarly, if $q \in B'$, then either $q \to X'$ or $X' \to q$.

If the conclusion of the lemma does not hold, then there must exist nodes $p \in A$ and $q \in B'$ such that (i) $p \to X$ and $q \to X'$ or (ii) $X \to p$ and $X' \to q$. If case (i) holds, then $\alpha(p) \to Z_o$ and $r \to S_o$ where $r = \alpha^{-1}(q)$; in particular $r \to p$ and $\alpha(p) \to \alpha(r) = q$. But this contradicts the definition of α , and case (ii) is also impossible

by symmetry. This completes the proof of the lemma.

Lemma 3.2. Suppose that $R \circ S = W \circ Z$. Then

a) for any copy S_o , there exist at most two copies Z_i such that $\alpha(S_o) \cap Z_i \neq \phi$, $\alpha(S_o) \nsubseteq Z_i$ and $Z_i \nsubseteq \alpha(S_o)$; and

b) for any copy Z_o , there exist at most two copies S_i such that $\alpha(S_i) \cap Z_o \neq \phi$, $\alpha(S_i) \nsubseteq Z_o$ and $Z_o \nsubseteq \alpha(S_i)$.

Proof. If there are at least three copies Z_i with the properties described in (a) , let $X_i = S_o \cap \alpha^{-1}(Z_i)$. If we apply Lemma 3.1 to each of the X_i's in turn we find that some two of them must dominate each other. This is impossible. Conclusion (b) follows by symmetry.

Lemma 3.3. If $R \circ S = W \circ Z$ where $|S| \leq |Z|$, then exactly one of the following alternatives holds for each copy S_o .

a) There exists a copy Z_o such that $\alpha(S_o) \subseteq Z_o$, or

b) there exist two copies Z_1 and Z_2 and tournaments Y,X,U and V such that $S_o = Y + X$, $Z_1 = U + \alpha(Y)$, $Z_2 = \alpha(X) + V$, and $Z_1 \rightarrow Z_2$.

Proof. For any copy Z_i such that $\alpha(S_o) \cap Z_i \neq \phi$ either $\alpha(S_o) \subseteq Z_i$ or $\alpha(S_o) \nsubseteq Z_i$ and $Z_i \nsubseteq \alpha(S_o)$, since $|S_o| \leq |Z_i|$. The result now follows from Lemmas 3.2 (a) and 3.1.

Lemma 3.4. If $R \circ S = W \circ Z$ where $|S| \leq |Z|$, then exactly one of the following alternatives holds for each copy Z_o .

a) There exist copies $S_1, S_2, \cdots, S_t$ such that $Z_o = \alpha(S_1 \cup S_2 \cup \cdots \cup S_t)$, or

b) there exist copies $S_1, S_2, \cdots, S_t$ and subtournaments X and Y of S_1 and S_t such that

$$Z_o = \alpha(X \cup S_2 \cup \cdots \cup S_{t-1} \cup Y) = \alpha(X) + \alpha(S_2 \cup \cdots \cup S_{t-1}) + \alpha(Y) \text{ , or}$$

c) there exist copies $S_1, \cdots, S_t$ and a subtournament X of S_1 such that

$$Z_o = \alpha(X \cup S_2 \cup \cdots \cup S_t) = \alpha(X) + \alpha(S_2 \cup \cdots \cup S_t) \quad \text{, or}$$

d) there exist copies $S_1, \cdots, S_t$ and a subtournament Y of S_t such that

$$Z_o = \alpha(S_1 \cup \cdots \cup S_{t-1} \cup Y) = \alpha(S_1 \cup \cdots \cup S_{t-1}) + \alpha(Y) \quad .$$

<u>Proof</u>. For any copy S_i such that $\alpha(S_i) \cap Z_o \neq \phi$, either $\alpha(S_i) \subseteq Z_o$ or $\alpha(S_i) \nsubseteq Z_o$ and $Z_o \nsubseteq \alpha(S_i)$. The result now follows from Lemmas 3.2.(b) and 3.1.

Notice that for any set $\{S_i | i \in I\}$ of copies of S in $R \circ S$, there exists a subtournament C_I of R such that $\bigcup_{i \in I} S_i = C_I \circ S$; this follows from the definition of $R \circ S$.

§4. <u>Chains</u>

If $R \circ S = W \circ Z$, then we shall say that the copy Z_o of Z in $W \circ Z$ is of type (00) , (11) , (10) or (01) according as alternative (a), (b), (c), or (d) holds in Lemma 3.4.

Suppose some copy, say Z_1, is of type (01) ; then there exists some copy S_1 (changing our notation from before) such that $S_1 = Y_1 + X_1$ and $Z_1 = V_1 + Y_1'$ where $\alpha(Y_1) = Y_1'$, for some subtournaments Y_1, X_1, V_1 and Y_1' (this follows from Lemmas 3.4 (d) and 3.1) . But then, by Lemmas 3.3 and 3.1, there must exist a copy Z_2 such that $Z_2 = X_1' + V_2$ where $\alpha(X_1) = X_1'$; furthermore $Z_1 \to Z_2$. Now Z_2 is either of type (10) or type (11) . If it is of type (11) then we can repeat this argument and find copies S_2 and Z_3 such that $S_2 = Y_2 + X_2$, $Z_2 = X_1' + U_1 + Y_2'$ and $Z_3 = X_2' + V_2$, where $\alpha(Y_2) = Y_2'$ and $\alpha(X_2) = X_2'$; furthermore $Z_2 \to Z_3$. If Z_3 is of type (11) , we can repeat the argument again. Eventually the process will terminate when we reach some copy Z_m of type (10) . The copies $Z_1, Z_2, \cdots, Z_m$ form what we shall call an <u>open chain</u> in $W \circ Z$ and we shall call $S_1, \cdots, S_{m-1}$ the <u>corresponding copies</u> in $R \circ S$. If the copy Z_1 with which we started had been of type (11) then we could have applied this construction in both directions. This would also have led to an open chain unless the construction in both directions eventually led to the same copy. In this latter case (which cannot occur, as we shall show presently) we would say that the copies of Z involved form a <u>closed chain</u>. Thus all the copies of Z of type (11) , (01) , and (10) can be partitioned in a natural way into a collection of disjoint chains. If $s = |S|$ and $z = |Z|$, let $z = qs + r$, where $0 \leq r < s$; as usual (a,b) denotes the greatest common divisor of a and b.

<u>Lemma 4.1</u>. Suppose that $R \circ S = W \circ Z$, where $|S| \leq |Z|$. If the distinct copies $Z_1, \cdots, Z_m$ form a chain in $W \circ Z$, then $m = \dfrac{s}{(s,r)}$ and $Z_1 \cup \cdots \cup Z_m = T_m \circ Z$.

Proof. Let $S_i = Y_i + X_i$ denote the corresponding copies in $R \circ S$; we use the same notation as before so that $Z_1 = V_1 + Y_1'$, $Z_i = X_{i-1}' + U_i + Y_i'$ for $i = 2,3,\cdots,m-1$, and $Z_m = X_{m-1}' + V_m$ where $\alpha(X_i) = X_i'$ and $\alpha(Y_i) = Y_i'$ for each i.

Suppose $Z_i \rightarrow Z_j$ where $1 \leq i < j \leq m-1$. Then $Y_i' \rightarrow Y_j'$ since $Y_i' \subseteq Z_i$ and $Y_j' \subseteq Z_j$. But then $\alpha^{-1}(Y_i') \rightarrow \alpha^{-1}(Y_j')$ or $Y_i \rightarrow Y_j$. Therefore $S_i \rightarrow S_j$ and, in particular, $Y_i \rightarrow X_j$. Consequently, $Y_i' \rightarrow X_j'$ and this implies that $Z_i \rightarrow Z_{j+1}$. Since we know already that $Z_i \rightarrow Z_{i+1}$ for $1 \leq i \leq m-1$ it follows by induction that each copy in a chain dominates all succeeding copies in the chain. Hence, closed chains do not exist (otherwise we would have the impossible situation that $Z_1 \rightarrow Z_m$ and $Z_m \rightarrow Z_1$) and each open chain can be expressed as $T_m \circ Z$ for a transitive tournament T_m. It remains to determine the value of m.

If $x_i = |X_i| = |X_i'|$ and $y_i = |Y_i| = |Y_i'|$ then $x_i + y_i = s$ for $1 \leq i \leq m-1$. Copies Z_1 and Z_m of the chain are of types (01) and (10) while the intermediate copies are of type (11); hence $y_1 = r$ and $x_i + y_{i+1} \equiv r \pmod{s}$ for $1 \leq i \leq m-2$. This implies that $y_i \equiv ir \pmod{s}$ for $1 \leq i \leq m-1$ and that m, the number of the copy of type (10) where the chain terminates, is the first positive integer such that $mr \equiv 0 \pmod{s}$. Therefore $m = \frac{s}{(s,r)}$ and the lemma is proved.

Lemma 4.2. Suppose that $R \circ S = W \circ Z$ where $|S| \leq |Z|$ and that the copies $Z_{11},\cdots,Z_{1m}$ and $Z_{21},\cdots,Z_{2m}$ form two chains. If $Z_{11} \rightarrow Z_{21}$, then $Z_{1i} \rightarrow Z_{2j}$ for $1 \leq i,j \leq m$.

Proof. When $t = 1,2$ let $S_{ti} = Y_{ti} + X_{ti}$ denote the corresponding copies in $R \circ S$ so that $Z_{t1} = V_{t1} + Y'_{t1}$, $Z_{ti} = X'_{t,i-1} + U_{ti} + Y'_{ti}$ for $2 \leq i \leq m-1$, and $Z_{tm} = X'_{t,m-1} + V_{tm}$; furthermore, $\alpha(X_{ti}) = X'_{ti}$ and $\alpha(Y_{ti}) = Y'_{ti}$ for each t and i.

Suppose $Z_{1i} \to Z_{2j}$ for some fixed values of i and j where $1 \leq i,j < m$. Then, $Y'_{1i} \to Y'_{2j}$ so $Y_{1i} \to Y_{2j}$. Since all the edges between S_{1i} and S_{2j} go the same way, this implies that $Y_{1i} \to X_{2j}$ and $X_{1i} \to Y_{2j}$. But then $Y'_{1i} \to X'_{2j}$ and $X'_{1i} \to Y'_{2j}$. This implies that $Z_{1i} \to Z_{2,j+1}$ and $Z_{1,i+1} \to Z_{2j}$. Since we may assume that $Z_{11} \to Z_{21}$ it now follows by induction that $Z_{1i} \to Z_{2j}$.

§5. Main Results

Theorem 1. Suppose that $R \circ S = W \circ Z$ and that $z = qs + r$ where $z = |Z|$, $s = |S|$, $q \geq 1$ and $0 \leq r < s$.

a) If $|S| = |Z|$, then $R = W$ and $S = Z$;

b) if $r = 0$, then $Z = Q \circ S$ for some tournament Q; and

c) if $r > 0$, then $R = V \circ T_{\frac{z}{(s,r)}}$, $S = T_{\frac{s}{c}} \circ C$, $W = V \circ T_{\frac{s}{(s,r)}}$ and

$Z = T_{\frac{z}{c}} \circ C$ for some tournaments V and C, where $c = |C|$.

Proof. If $r = 0$, there are no copies of Z in $W \circ Z$ of type (01) or (10); thus there are no chains in $W \circ Z$, in view of Lemma 4.1. Therefore, every copy of Z is of type (00). Conclusion (b) now follows from the remark after Lemma 3.4.

Conclusion (a) follows from (b) since if $|S| = |Z| = |Q \circ S|$ it must be that $|Q| = 1$, or that $Z = S \circ T_1 = S$. Furthermore, the mapping α between $R \circ S$ and $W \circ Z$ takes copies of S onto copies of Z when $S = Z$; consequently α defines an isomorphism between R and W .

If $r > 0$, there are no copies of type (00) in $W \circ Z$. Hence, all the copies of Z in $W \circ Z$ can be partitioned into disjoint open chains of length $m = \frac{s}{(s,r)}$ by Lemmas 3.4 and 4.1. It now follows from Lemmas 4.1 and 4.2 (and the definition of composition) that $W = V \circ T_m$ for some tournament V .

Let Z_1 and Z_m denote the first and last copies in some chain in $W \circ Z$ and let $S_1 = Y_1 + X_1$ and $S_{m-1} = Y_{m-1} + X_{m-1}$ denote the corresponding copies in $R \circ S$. It follows from the remark after Lemma 3.4 and the definitions of copies of type (01) and (10) that $Z_1 = (H \circ S) + Y_1'$ and $Z_m = X_{m-1}' + (G \circ S)$ for some tournaments H and G where $\alpha(Y_1) = Y_1'$ and $\alpha(X_{m-1}) = X_{m-1}'$. But $S_1 = S = S_{m-1}$ and $Z_1 = Z = Z_m$, so $Y_{m-1} + X_{m-1} = Y_1 + X_1$ and

$$X_{m-1} + [G \circ (Y_{m-1} + X_{m-1})] = [H \circ (Y_1 + X_1)] + Y_1 \quad .$$

Appealing to Lemma 2.8, we conclude that $S = T_k \circ C + T_j \circ C = T_{k+j} \circ C$ and $Z = T_j \circ C + [T_\ell \circ (T_{k+j} \circ C)] = T_{\ell(k+j)+j} \circ C$, for some tournaments T_k, T_j, T_ℓ and C where $T_k, T_j,$ and T_ℓ are transitive. If $c = |C|$, then it must be, since $s = |S|$ and $z = |Z|$, that $S = T_{s/c} \circ C$ and $Z = T_{z/c} \circ C$.

We know that each chain in $W \circ Z$ is of the form $T_m \circ Z$. The

inverse image in $R \circ S$ of each such chain is a union of copies of S, by the definition of a chain, and hence is of the form $D \circ S$ for some tournament D (see the remark after Lemma 3.4). Therefore,

$$D \circ T_{s/c} \circ C = D \circ S = T_m \circ Z = T_m \circ T_{z/c} \circ C = T_{mz/s} \circ T_{s/c} \circ C$$

so, by conclusion (a), it must be that $D = T_{mz/s} = T_{z/(s,r)}$.

All edges between any two distinct chains in $W \circ Z$ go the same way. Therefore, all edges between the inverse images in $R \circ S$ of any two such chains must go the same way. It follows from this and the result in the preceding paragraph that $R = Q \circ T_{z/(s,r)}$ for some tournament Q . Finally, since

$$Q \circ T_{z/(s,r)} \circ T_{s/c} \circ C = R \circ S = W \circ Z = V \circ T_{s/(s,r)} \circ T_{z/c} \circ C ,$$

it must be that $Q = V$ by conclusion (a) . This completes the proof of the theorem.

<u>Corollary 1.1</u>. If $A \circ X = B \circ X$, then $A = B$.

<u>Corollary 1.2</u>. If $X \circ A = X \circ B$, then $A = B$.

These cancellation laws follow directly from part (a) of Theorem 1 (another proof of these laws has been given by Reid [21]) .

We say that a non-trivial tournament R is <u>prime</u> if there do not exist non-trivial tournaments A and B such that $R = A \circ B$. Thus, the transitive tournament T_n is prime if and only if the integer n is prime. Every tournament can be expressed as the ordered composition of

prime tournaments but the decomposition need not be unique since, for example, $T_6 = T_2 \circ T_3 = T_3 \circ T_2$. The following results show, however, that if consecutive transitive factors are grouped together, then the decomposition is essentially unique.

Lemma 5.1. Suppose that $R \circ S = W \circ Z$ and that $z = qs + r$ where $z = |Z|$, $s = |S|$, $q \geq 1$ and $0 \leq r < s$. Furthermore, suppose that each of the non-trivial tournaments S and Z is either transitive or prime.

a) If $|S| = |Z|$, then $R = W$ and $S = Z$;

b) if $r = 0$ and $|S| < |Z|$, then $R = W \circ T_q$, $S = T_s$ and $Z = T_q \circ T_s = T_z$;

c) if $r > 0$, then $R = V \circ T_{z/(s,r)}$, $S = T_s$, $W = V \circ T_{s/(s,r)}$ and $Z = T_z$ for some tournament V .

This follows directly from Theorem 1 and the fact that if $X \circ Y$ is transitive then X and Y must both be transitive.

Theorem 2. Suppose that $S_1 \circ S_2 \circ \cdots \circ S_m = Z_1 \circ Z_2 \circ \cdots \circ Z_n$ where each of the factors S_i and Z_j is a non-trivial transitive or prime tournament. If no two consecutive factors S_i and S_{i+1} or Z_j and Z_{j+1} are both transitive, then $m = n$ and $S_i = Z_i$ for $1 \leq i \leq n$.

Proof. If $S_m \neq Z_n$ then we may suppose, by Lemma 5.1, that S_m and Z_n are both transitive and that $U \circ T_v = S_1 \circ \cdots \circ S_{m-1}$ for some tournament U and some integer v , where $v > 1$. But then S_{m-1} must be

transitive also, by Lemma 5.1, and this contradicts the fact that S_m and S_{m-1} cannot both be transitive. Therefore $S_m = Z_n$ and the result now follows by induction.

Lemma 5.2. If $T_a \circ D \circ T_b = T_\alpha \circ D \circ T_\beta$, then either D is transitive or $a = \alpha$ and $b = \beta$.

Proof. If D is not transitive, then $D = T_p \circ D_1 \circ \cdots \circ D_\ell \circ T_q$ where $p,q,\ell \geq 1$, the non-trivial tournaments $D_1, \cdots, D_\ell$ are either transitive or prime, no two consecutive tournaments D_i and D_{i+1} are both transitive, and neither D_1 nor D_ℓ is transitive. Therefore,

$$T_a \circ T_p \circ D_1 \circ \cdots \circ D_\ell \circ T_q \circ T_b = T_\alpha \circ T_p \circ D_1 \circ \cdots \circ D_\ell \circ T_q \circ T_\beta .$$

It follows from Theorem 2 that $T_a \circ T_p = T_\alpha \circ T_p$ and, hence, that $a = \alpha$; similarly $b = \beta$ and the lemma is proved.

Let $H^2 = H \circ H$ and $H^n = H \circ (H^{n-1})$ for $n = 3,4,\cdots$.

Theorem 3. If $Z \circ S = S \circ Z$ for non-trivial tournaments S and Z , then S and Z are both transitive or $S = H^u$ and $Z = H^v$ for some tournament H .

Proof. Suppose that $z = qs + r$ where $s = |S| \leq z = |Z|$ and $0 \leq r < s$. If $s = z$ then $S = Z$ by Theorem 1(a). If $s < z$ and $r = 0$ then $Z = C \circ S$ for some non-trivial tournament C ; hence $C \circ S = S \circ C$ by Corollary 1.1 and the result for this case now follows by induction on $m = \max(|S|,|Z|)$.

If $s < z$ and $r > 0$, then $Z = V \circ T_{z/(s,r)} = T_{z/c} \circ C$ and $S = T_{s/c} \circ C = V \circ T_{s/(s,r)}$ for some tournaments V and C by Theorem 1 (c); therefore, $T_{z/c} \circ C \circ V \circ T_{s/(s,r)} = Z \circ S = S \circ Z = T_{s/c} \circ C \circ V \circ T_{z/(s,r)}$. Since $s \neq z$, it follows from Lemma 5.2 that $C \circ V$ is transitive. But then C and V must both be transitive and, consequently, both S and Z are transitive. This completes the proof of the theorem.

§6. Almost All Tournaments are Prime

We say that a tournament is composite if it is not prime. Our object in this section is to show that the ratio of the number of nonisomorphic composite tournaments with n nodes to the number of nonisomorphic tournaments with n nodes tends to zero as n tends to infinity. We also obtain a recursion formula for the number of non-transitive composite tournaments with n nodes, using the following notation (the independent variable refers to the number of nodes in the class of tournaments involved):

$f(n)$ = the number of nonisomorphic tournaments,

$c(n)$ = the number of nonisomorphic composite tournaments,

$c^*(n)$ = the number of nonisomorphic nontransitive composite tournaments,

$p(n)$ = the number of nonisomorphic prime tournaments,

$q(n)$ = the number of nonisomorphic nontransitive prime tournaments, and

$g(n)$ = the number of nonisomorphic tournaments whose decomposition does not end in a transitive factor (we define $g(1) = 1$).

Let d be a divisor of n; the number of ways of forming a composite tournament with n nodes whose last prime factor with d nodes is not transitive is equal to $q(d)f(\frac{n}{d})$. Consequently, the number

of composite tournaments whose decomposition does not end in a transitive factor is given by $\sum\limits_{\substack{d|n \\ d\neq 1,n}} q(d)f(\frac{n}{d})$. The number of tournaments whose decomposition does end in a transitive factor is equal to $f(n)-g(n)$; one of these is the transitive tournament with n nodes. Therefore, the number of non-transitive composite tournaments with n nodes is given by

$$c^*(n) = \sum_{\substack{d|n \\ d\neq 1,n}} q(d)f(\tfrac{n}{d}) + f(n) - g(n) - 1 .$$

Since we may express any non-transitive tournament T in the form $T=A\circ B$ where B is transitive and $A = A_1\circ A_2\circ\cdots\circ A_k$ where A_k is not transitive, it follows that $f(n) = \sum\limits_{d|n} g(\frac{n}{d}) = \sum\limits_{d|n} g(d)$. Then if μ denotes the Möbius function, we have

$$g(n) = \sum_{d|n} \mu(d)f(\tfrac{n}{d}) = f(n) + \mu(n) + \sum_{\substack{d|n \\ d\neq 1,n}} \mu(d)f(\tfrac{n}{d}) ,$$

so that

$$\begin{aligned} c^*(n) &= \sum_{\substack{d|n \\ d\neq 1,n}} q(d)f(\tfrac{n}{d}) - \sum_{\substack{d|n \\ d\neq 1,n}} \mu(d)f(\tfrac{n}{d}) - \mu(n) - 1 \\ &= \sum_{\substack{d|n \\ d\neq 1,n}} [q(d) - \mu(d)]\, f(\tfrac{n}{d}) - \mu(n) - 1 . \end{aligned}$$

The number $c(n)$ of composite tournaments with n nodes, for $1 \leq n \leq 12$, is given in Table 1 (the first eight values of $f(n)$ where given by Davis;

see Moon [17; p. 87]) .

Now certainly $c(n) \leq \sum_{\substack{d|n \\ d\neq 1,n}} p(d)f(\frac{n}{d})$; also $f(\frac{n}{d}) \leq f([\frac{n}{2}])$ where d is not equal to 1 or n , and clearly $p(d) \leq f(d)$. Therefore,

$$\begin{aligned} c(n) &\leq \textstyle\sum' p(d)f(\frac{n}{d}) \leq \sum' p(d)f([\frac{n}{2}]) \\ &\leq \textstyle\sum' f(d)f([\frac{n}{2}]) \leq \sum' f([\frac{n}{2}])f([\frac{n}{2}]) \\ &\leq n\ f([\tfrac{n}{2}])f([\tfrac{n}{2}]) \ , \end{aligned}$$

where the sums are over all divisors of n other than 1 and n . Hence,

$$\frac{c(n)}{f(n)} \leq \frac{n\ f([\frac{n}{2}])\ f([\frac{n}{2}])}{f(n)} \ ,$$

and it follows from Stirling's formula and the fact that $f(n) \sim \frac{2^{\binom{n}{2}}}{n!}$ (see Moon [17, p. 88]) that the right hand side of the above inequality is less than a constant times $\sqrt{n}\ 2^{n-n^2/4}$ for all sufficiently large n ; this tends to zero as n tends to infinity.

§7. The Group of the Composition of Two Tournaments.

If T is any tournament, then an automorphism of T is an isomorphism from T onto itself. The set of automorphisms of T forms a group, called the (automorphism) group of T , denoted by $G(T)$.

n	f(n)	c(n)
1	1	0
2	1	1
3	2	0
4	4	1
5	12	0
6	56	3
7	456	0
8	6,880	7
9	191,536	4
10	9,733,056	23
11	903,753,248	0
12	154,108,311,168	122

TABLE 1

More will be said concerning the group of a tournament in Chapter 3; our purpose in this section is to determine the group $G(A \circ B)$ of the composition of two tournaments A and B, given $G(A)$ and $G(B)$. Sabidussi [24] and Hemminger [13] have considered the analogous problem for ordinary graphs.

Let R and G denote two permutation groups acting on sets M and N respectively, and let $r = |R|$, $g = |G|$, $m = |M|$ and $n = |N|$. The <u>composition</u> (or wreath product) of G by R is the group $R \circ G$ of order rg^m and degree mn consisting of all permutations α of

$$M \times N = \{(x,y) \mid x \in M , y \in N\}$$

of the form

$$\alpha(x,y) = (f(x) , h_x(y)) ,$$

where f is any element of R and h_x, for each x, is any element of G. If the elements of $M \times N$ are arranged in a matrix so that the rows and columns correspond to the elements of M and N respectively, then $R \circ G$ is the group of permutations obtained by permuting the objects in each row according to some element of G (not necessarily the same element for every row) and then permuting the rows themselves according to some element of R.

We obtain the following theorem, due to Alspach, Goldberg and Moon [1], as a simple consequence of the results in §5.

<u>Theorem 4</u>. $G(A \circ B) = G(A) \circ G(B)$.

<u>Proof</u>. It is not difficult to see that $G(A) \circ G(B)$ is a subgroup of $G(A \circ B)$. The authors observed that, to show the groups are identical, it suffices to prove that if a permutation α of $G(A \circ B)$ takes any node of a copy B_i of B to a node of a copy B_j of B , then α takes every node of B_i into B_j .

But this is immediate in view of Theorem 1, since there are no chains in $A \circ B$. Hence every copy of B is of type (00) . This means that every automorphism α takes copies of B onto copies of B , and this completes the proof of Theorem 4.

CHAPTER 2

Isomorphisms of Tournaments

§1. Introduction

Two graphs G and G' are <u>isomorphic</u> if there exists a one-to-one mapping ϕ between their nodes such that nodes x and y of G are joined by an edge if and only if $\phi(x)$ and $\phi(y)$ are joined by an edge in G'. If G and G' are related by a certain mapping between their edges, one may ask for conditions under which G and G' are isomorphic. H. Whitney [26] proved that if G and G' are 3-vertex connected and circuit isomorphic, then G and G' are isomorphic (for a simplified proof of this result, see Ore [18; p. 245]). More recently, Halin and Jung [11] have generalized Whitney's result and extended it to infinite graphs. Our object in this chapter is to obtain a theorem of this type for tournaments.

§2. Preliminary Results

Two tournaments T_n and T_n' are <u>anti-isomorphic</u> if they can be made isomorphic by reversing the orientation of every edge of one of them. Let ϕ denote a bijection between the edges of two tournaments T_n and T_n'. We say that T_n and T_n' are <u>h-cycle isomorphic</u> (with respect to ϕ) if any set of h edges in one tournament form a cycle if and only if the corresponding edges in the other tournament form a cycle (but not necessarily in the same order). In §3, we will prove the

following result.

Theorem. If two irreducible tournaments T_n and T'_n are 3-cycle and 4-cycle isomorphic with respect to ϕ, then they are either isomorphic or anti-isomorphic.

In what follows, we will denote an edge of unspecified orientation joining nodes i and j by ij. A nontrivial subtournament M of T_n has property $P(\phi)$ if ϕ induces an isomorphism or an anti-isomorphism between M and some subtournament N of T'_n, that is, if there exists a bijection α_M between the nodes of M and N such that

(i) $\phi(ij) = \alpha_M(i)\alpha_M(j)$ for all $i,j \in M$, and either

(ii) $i \to j \Longleftrightarrow \alpha_M(i) \to \alpha_M(j)$ for all $i,j \in M$, or

(iii) $i \to j \Longleftrightarrow \alpha_M(j) \to \alpha_M(i)$ for all $i,j \in M$.

An edge is good or bad for M with respect to ϕ and α_M according as (i) does or does not hold for that edge; M almost has property $P(\phi)$ if (ii) or (iii) holds for α_M and if M contains exactly one bad edge with respect to ϕ and α_M. For convenience, if α is any bijection from the nodes of M onto the nodes of N, we will say that ϕ and α are compatible on an edge ij if $\phi(ij) = \alpha(i)\alpha(j)$.

Lemma 2.1. Let α_R and α_S denote bijections of the nodes of subtournaments R and S of T_n onto the nodes of subtournaments U and V of T'_n, respectively. Suppose that ϕ is incompatible with α_R and

and α_S on at most one edge of R and S, respectively. If at most one of these exceptional edges is in $R \cap S$ and $|R \cap S| \geq 3$ then α_R and α_S are identical on $R \cap S$. Suppose, in addition, that α_R defines an isomorphism or anti-isomorphism between R and U, and α_S either defines an isomorphism or anti-isomorphism between S and V or would if the orientation of some edge $\alpha_S(r)\alpha_S(s)$ where $rs \in R \cap S$ were reversed. If α is defined by $\alpha(x) = \alpha_R(x)$ if $x \in R$ and $\alpha(x) = \alpha_S(x)$ if $x \in S$, then α is a bijection of the nodes of $R \cup S$ onto the nodes of $U \cup V$ and either

$$i \to j \iff \alpha(i) \to \alpha(j) \quad \text{for all } i,j \in R \text{ and all } i,j \in S \quad \text{or}$$

$$i \to j \iff \alpha(j) \to \alpha(i) \quad \text{for all } i,j \in R \text{ and all } i,j \in S \quad .$$

Proof. It follows from the hypotheses that there exists at least one node c in $R \cap S$ such that no edge in $R \cap S$ which is incident with c is bad for either R or S. Let ca and cb be two distinct such edges. Then

$$\phi(ca) = \alpha_R(c)\alpha_R(a) = \alpha_S(c)\alpha_S(a) \quad , \quad \text{and}$$

$$\phi(cb) = \alpha_R(c)\alpha_R(b) = \alpha_S(c)\alpha_S(b) \quad .$$

If $\alpha_R(c) \neq \alpha_S(c)$, then it must be that $\alpha_R(c) = \alpha_S(a)$ and $\alpha_R(c) = \alpha_S(b)$; but then $\alpha_S(a) = \alpha_S(b)$, a contradiction since $a \neq b$. Therefore, $\alpha_R(c) = \alpha_S(c)$ and it follows that $\alpha_R(a) = \alpha_S(a)$ and $\alpha_R(b) = \alpha_S(b)$. Hence $\alpha_R = \alpha_S$ on $R \cap S$.

If $\alpha(y) = \alpha(z)$ for some $y \in R-S$ and $z \in S-R$, then for some node w in $R \cap S$ it must be that ϕ is compatible with α_R and α_S on yw and zw, respectively; but then

$$\phi(yw) = \alpha(y)\alpha(w) = \alpha(z)\alpha(w) = \phi(zw) \quad ,$$

a contradiction. Hence α is the required bijection, and the remainder of the conclusion follows from the definition of isomorphism and anti-isomorphism. This completes the proof of the lemma.

Lemma 2.2. If T_{h+1} is irreducible and $h+1 \geq 4$, then there exist two irreducible subtournaments of T_{h+1}, both with h nodes, having $h-1$ nodes in common.

Proof. Since T_{h+1} is irreducible, it contains an h-cycle [17, p. 6] which determines an irreducible subtournament H_1 with h nodes. The node of T_{h+1} not contained in H_1 is itself contained in an h-cycle [17; p. 6] and hence in an irreducible subtournament H_2 of T_{h+1} with h nodes and with $h-1$ nodes in common with H_1. This proves the lemma.

Lemma 2.3. Let $\overrightarrow{pq}$ belong to the irreducible tournament T_{h+1} and suppose that $\overrightarrow{pq}$ is not contained in any irreducible subtournament of T_{h+1} with at most h nodes. Then the nodes of T_{h+1} may be labelled a_i (where $1 \leq i \leq h+1$ and $p = a_1$, $q = a_{h+1}$) in such a way that

$$a_i \to a_j \quad \text{for} \quad 1 \leq i < j \leq h+1 \quad \text{except that} \quad a_{i+1} \to a_i \;.$$

Proof. Since T_{h+1} is irreducible (or strongly connected, see Chapter 1, §2) there must exist a path in T_{h+1} from q to p. This path, together with $\overrightarrow{pq}$ determines a cycle and hence an irreducible subtournament of T_{h+1}. It follows from the hypotheses that this subtournament must contain $h+1$ nodes - in other words, $\overrightarrow{pq}$ is contained in a spanning cycle of T_{h+1}. The result is now a direct consequence of the previously noted fact that an ℓ-cycle in T_{h+1} determines an irreducible subtournament of T_{h+1} with ℓ nodes.

Let T_n and T'_n be irreducible. If $h \geq 4$ and if all nontrivial irreducible subtournaments of T_n and T'_n with at most h nodes have properties $P(\phi)$ and $P(\phi^{-1})$, respectively, then we say that T_n and T'_n are <u>h-equivalent</u> with respect to ϕ. The following lemmas are central to the proof of one of our main results in §3.

Lemma 2.4. Let T_n and T'_n be irreducible and h-equivalent with respect to ϕ, where $h \geq 4$, and let T be an irreducible subtournament of T_n with $h+1$ nodes. If T does not have property $P(\phi)$ then it almost has property $P(\phi)$ and is a tournament of the type described in Lemma 2.3 with $\overrightarrow{pq}$ as its bad edge.

Proof. It follows from Lemma 2.2 that T contains distinct nodes p and q and irreducible subtournaments A and B, both with h nodes, such that $p \in A$, $p \notin B$, $q \in B$, $q \notin A$; we may suppose that $p \rightarrow q$. The subtournaments A and B have property $P(\phi)$ and since $|A \cap B| \geq 3$ we may apply Lemma 2.1 to A and B to conclude that ϕ induces a bijection α from the nodes of T onto the nodes of a subtournament

T' of T'_n which is compatible with ϕ on all edges of T with the possible exception of $\overrightarrow{pq}$. Moreover, α defines an isomorphism or an anti-isomorphism between T and T' or would if the orientation of the edge $\alpha(p)\alpha(q)$ were reversed.

If $\overrightarrow{pq}$ is contained in an irreducible subtournament Q of T with at most h nodes, then Q has property $P(\phi)$ and it follows from Lemma 2.1 (with $R = Q$ and $S = T$) that ϕ and α are compatible on $\overrightarrow{pq}$ and hence T has property $P(\phi)$.

If $\overrightarrow{pq}$ is not contained in any irreducible subtournament of T with at most h nodes, then T is a tournament of the type described in Lemma 2.3, and so is T' (or it would be if the orientation of the edge $\alpha(p)\alpha(q)$ were reversed). But the orientation of $\alpha(p)\alpha(q)$ must be consistent with the orientation of the other edges of T' with respect to T since otherwise $\alpha(p)\alpha(q)$ would be contained in a 3-cycle in T' whose image under ϕ^{-1} is not a 3-cycle in T . It follows that α is an isomorphism or an anti-isomorphism from T to T' and this suffices to complete the proof of Lemma 2.4.

<u>Lemma 2.5</u>. Let T_n and T'_n be irreducible and h-equivalent with respect to ϕ , where $h \geq 4$, and let T be an irreducible subtournament of T_n with $h+1$ nodes that almost has property $P(\phi)$ with $\overrightarrow{pq}$ as its bad edge. Then $\overrightarrow{pq}$ is not contained in any irreducible subtournament of T_n with at most h nodes.

<u>Proof</u>. We will prove the contrapositive of this statement by considering various possibilities for $\overrightarrow{pq}$. Notice that it follows from Lemma

2.4 that T is a tournament of the type described in Lemma 2.3, so that $\overrightarrow{pq}$ is not contained in any irreducible subtournament of T with at most h nodes.

Case 1: Suppose $\overrightarrow{pq}$ is contained in an irreducible subtournament S of T_n with at most h nodes which contains at least one other node of T. Since S has property $P(\phi)$ by hypothesis and T almost has property $P(\phi)$, and since $|S \cap T| \geq 3$ we may apply Lemma 2.1 to S and T to conclude that the bijections induced by ϕ on S and on T must be identical on $S \cap T$; this contradicts the fact that $\overrightarrow{pq}$ was bad for T.

Case 2: Suppose that $\overrightarrow{pq}$ is contained in an irreducible subtournament S of T_n with at most $h-1$ nodes which contains no other nodes of T. Since $h+1 \geq 5$, there exists at least one node r of T such that $p \to r$ and $r \to q$. The tournament R with at most h nodes obtained from S by adjoining the node r is irreducible; thus, we may apply the argument in Case 1 to R to again contradict the fact that $\overrightarrow{pq}$ was bad for T.

Case 3: Suppose that $\overrightarrow{pq}$ is contained in an irreducible subtournament S of T_n with h nodes which contains no other nodes of T. The irreducible tournament R with $h+1$ nodes obtained from S by adjoining a node r of T such that $p \to r$ and $r \to q$ has or almost has property $P(\phi)$ by Lemma 2.4. We show that $\overrightarrow{pr}$ is a good edge for R. To see this, observe that $\overrightarrow{pr}$ is contained in an irreducible subtournament of T, namely the tournament U defined by $\overrightarrow{pr}$ and the path in T from r

to p (note that U does not contain any other nodes of R). Thus we may apply Case 2 to the edge $\overrightarrow{pr}$ and the tournaments R and U (since $|U| \leq h-1$) to conclude that $\overrightarrow{pr}$ is a good edge for R , and we find, in a similar way, that $\overrightarrow{rq}$ is a good edge for R . The tournament S has property $P(\phi)$ and since $\overrightarrow{pq}$ is a good edge for S , it follows upon applying Lemma 2.1 to S and R that $\overrightarrow{pq}$ is good for R . Hence, every edge in $T \cap R$ is good for R and we may apply Lemma 2.1 to T and R to obtain, as before, a contradiction to the fact that $\overrightarrow{pq}$ was bad for T . This exhausts the possibilities for $\overrightarrow{pq}$ and completes the proof of Lemma 2.5.

Lemma 2.6. Let T_n and T'_n be irreducible and h-equivalent with respect to ϕ , where $h \geq 4$, and let T be an irreducible subtournament of T_n with $h+1$ nodes that almost has property $P(\phi)$ with $\overrightarrow{pq}$ as its bad edge. If $\overrightarrow{pq}$ is contained in another irreducible subtournament U of T_n with $h+1$ nodes, then

(i) U almost has property $P(\phi)$ with $\overrightarrow{pq}$ as its bad edge;

(ii) if α_U and α_T are the bijections induced by ϕ on U and T then the bijection α defined by

$$\alpha(x) = \begin{cases} \alpha_U(x) & \text{for} \quad x \in U \\ \alpha_T(x) & \text{for} \quad x \in T \end{cases}$$

is compatible with ϕ on all edges of U and T except $\overrightarrow{pq}$;

(iii) either $i \to j \iff \alpha(i) \to \alpha(j)$ for all $i,j \in U$ and for all $i,j \in T$, or $i \to j \iff \alpha(j) \to \alpha(i)$ for all $i,j \in U$ and for all $i,j \in T$.

<u>Proof</u>. As before, it follows from Lemma 2.4 that T is a tournament of the type described in Lemma 2.3.

<u>Case 1</u>: U contains at least one other node of T. It must be that U has or almost has property $P(\phi)$, by Lemma 2.4. But if there were no bad edges for U which belonged to T, then by Lemma 2.1, $\overrightarrow{pq}$ would be a good edge for T and this is a contradiction. Therefore, U almost has property $P(\phi)$; since all edges of $U \cap T$ except $\overrightarrow{pq}$ belong to irreducible subtournaments of T with at most h nodes, it follows from Lemma 2.5 that these edges are good for U and so $\overrightarrow{pq}$ is the bad edge for U. If we apply Lemma 2.1 to U and T, the result follows.

<u>Case 2</u>: U does not contain any other nodes of T. Since $\overrightarrow{pq}$ is bad for T, it follows from Lemmas 2.5 and 2.3 that U is a tournament of the type described in Lemma 2.3 and by Lemma 2.4 it follows that U has or almost has property $P(\phi)$. If we label the nodes of T and U by a_i and b_i, respectively, as in Lemma 2.3, consider the subtournament Q of T_n defined by $q = a_{h+1} = b_{h+1}$, a_h, b_h, a_{h-1} and b_{h-1} which is clearly irreducible and has three nodes in common with U and with T. Moreover, Q has property $P(\phi)$ except, possibly, when $h = 4$ in which case it almost has property $P(\phi)$ and its bad edge does not belong to $Q \cap T$ or $Q \cap U$ since the edges in these subtournaments of Q belong to 3-cycles. If α_Q denotes the node-bijection induced by ϕ on Q, then it follows from Lemma 2.1 that α_Q is identical with α_T on $Q \cap T$ and with α_U on $Q \cap U$. Therefore, $\alpha_U(q) = \alpha_Q(q) = \alpha_T(q)$ and we find, in a similar way, that $\alpha_U(p) = \alpha_T(p)$. Hence, it must be

that $\alpha_Q(p)\alpha_Q(q) = \alpha_U(p)\alpha_U(q) \neq \phi(pq)$. This implies that U almost has property $P(\phi)$ with $\overrightarrow{pq}$ as its bad edge. This suffices to complete the proof of the lemma.

§3. Main Results

Theorem 3.1. If T_n and T'_n are irreducible and h-equivalent with respect to ϕ , where $h \geq 4$, then there exists a bijection ψ from the edges of T_n onto the edges of T'_n such that T_n and T'_n are (h+1)-equivalent with respect to ψ ; furthermore, ϕ and ψ agree on all edges contained in irreducible subtournaments with at most h nodes.

Proof. Lemmas 2.5 and 2.6 imply that the edges of T_n may be partitioned into sets A , B and C where A is the set of edges which are contained in irreducible subtournaments with at most $h+1$ nodes that have property $P(\phi)$, B is the set of bad edges for irreducible subtournaments with $h+1$ nodes that almost have property $P(\phi)$, and C is the set of edges which are not contained in any irreducible subtournament of T_n with at most $h+1$ nodes. The edges of T'_n may be similarly partitioned into corresponding sets A' , B' and C' , respectively; it follows from the definition of A and A' that $\phi(A) = A'$.

We now define a bijection ψ from the edges of T_n onto the edges of T'_n as follows. If $ij \in A$, then let $\psi(ij) = \phi(ij)$. If $pq \in B$, then let $\psi(pq) = \alpha_M(p)\alpha_M(q)$ where α_M is the node-bijection induced by ϕ on any irreducible subtournament M of T_n with $h+1$ nodes that almost has property $P(\phi)$ with pq as its bad edge. Notice

that in view of Lemma 2.6 applied to T_n and to T'_n, the same bijection is obtained independently of which subtournament M is chosen which has pq as its bad edge. Finally, define ψ on edges of C according to some bijection which must certainly exist between C and C'. That T_n and T'_n are now $(h+1)$-equivalent with respect to ψ now follows from Lemmas 2.4 and 2.6 and the definitions of h- and $(h+1)$-equivalence, and suffices to complete the proof of the theorem.

We now use Theorem 3.1 to prove our main result by induction.

Theorem 3.2. If two irreducible tournaments T_n and T'_n are 3-cycle and 4-cycle isomorphic with respect to ϕ, then they are either isomorphic or anti-isomorphic.

Proof. We first show that T_n and T'_n are 4-equivalent with respect to ϕ. Notice that it follows from the definition of ϕ that any irreducible subtournament of T_n with three nodes has property $P(\phi)$. Let e' denote the image of the edge e under ϕ and let the ordered m-tuple $(a,b,\cdots,k)$ mean that the edges $a,b,\cdots,k$ form an m-cycle where a is followed by b, b is followed by c, $\cdots$, and k is followed by a. Let T be an irreducible subtournament of T_n with four nodes; it consists of a 4-cycle $(1,2,3,4)$ and two 3-cycles $(3,4,5)$ and $(2,3,6)$.

By hypothesis, $(3,4,5)$ implies $(3',4',5')$ or $(5',4',3')$, and $(2,3,6)$ implies $(2',3',6')$ or $(6',3',2')$. Since the edges $\{1',2',3',4'\}$ form a 4-cycle in T'_n, it must be that $(2',3',6')$ implies

$(3',4',5')$ and $(6',3',2')$ implies $(5',4',3')$. It follows that $(2',3',6')$ implies $(2',3',4',1')$ and $(6',3',2')$ implies $(4',3',2',1')$.

Suppose the nodes of the 4-cycle in T are labelled consecutively by a,b,c,d where a is the initial node of edge 1 , and suppose the nodes of the corresponding 4-cycle in T'_n are similarly labelled by a',b',c',d' where a' is the initial node of edge $1'$. If $(2',3',6')$, then the mapping α which takes a,b,c,d into a',b',c',d' , respectively, is an isomorphism induced by ϕ ; if $(6',3',2')$, then the mapping α which takes a,b,c,d into b',a',d',c' , respectively, is an anti-isomorphism induced by ϕ . It follows that in either case, T has property $P(\phi)$.

If $n \geq 4$, then it follows by induction from Theorem 3.1 that T_n and T'_n are n-equivalent with respect to some edge-bijection ψ . The theorem now follows upon observing that if T_n is irreducible, then T_n has property $P(\psi)$.

Theorem 3.2 does not necessarily hold if T_n and T'_n are not irreducible. For example, let T_1 and T_4 denote irreducible tournaments with one and four nodes, and let $T_6 = T_1 + T_1 + T_4$ and $T'_6 = T_1 + T_4 + T_1$. One can define an edge-bijection ϕ from T_6 to T'_6 such that T_6 and T'_6 are 3-cycle and 4-cycle isomorphic with respect to ϕ , but they are neither isomorphic nor anti-isomorphic.

In addition, Theorem 3.2 may not hold if T_n and T'_n are not 4-cycle isomorphic. For example, let T_1 denote the trivial tournament

with one node, let T_2 denote the reducible tournament with two nodes, and let T_4 denote the reducible tournament with four nodes consisting of one node which dominates each node of a 3-cycle. If we let T_7 and T'_7 be the irreducible tournaments obtained by letting $T_1 \to T_2 \to T_4 \to T_1$ and $T_1 \to T_4 \to T_2 \to T_1$, respectively, then one can define a 3-cycle isomorphism between these two tournaments (induced by an isomorphism between the copies of T_1 , T_2 and T_4) but they are neither 4-cycle isomorphic, nor isomorphic, nor anti-isomorphic.

CHAPTER 3

The Group of the Quadratic Residue Tournament

§1. Introduction

The (automorphism) group $G(T_n)$ of a tournament T_n was defined in Chapter 1, §7. In this chapter, we consider the group of a class of regular tournaments, that is, tournaments T_{2m+1} in which every node dominates exactly m other nodes. The groups of certain other specific graphs and tournaments have been considered, for example, in [1], [2], and [10]. References on the groups of graphs in general and tournaments in particular may be found in [18] and [17].

It is known [17] that there exist tournaments whose group is abstractly isomorphic to a given group H if and only if H has odd order; thus all tournament groups are solvable, by the Feit-Thompson Theorem [9].

Consider the Galois field $GF(q)$, where $q = p^n \equiv 3 \pmod 4$, that is, where n is odd and $p \equiv 3 \pmod 4$. Since -1 is a (quadratic) nonresidue in the field $GF(p)$ and n is odd, it follows that -1 is a nonresidue in $GF(q)$ [5 ; p. 45, §62]. This implies that if $a-b$ is a residue in $GF(q)$ then $b-a$ is a nonresidue in $GF(q)$.

If we now label $q = p^n$ nodes with the elements of the Galois field $GF(q)$ and let $a_i \to a_j$ if and only if $a_j - a_i$ is a square in

$GF(q)$, then the preceding observation guarantees that the resulting configuration will be a tournament when $q \equiv 3 \pmod 4$. It is clear that the tournament thus defined is regular; we call it the (quadratic) residue tournament R_q . Our main object in this chapter is to determine the group $G(R_q)$ of the residue tournament R_q .

§ 2. Preliminary Results

Finding $G(R_q)$ is equivalent to finding all permutations ϕ of the elements of $GF(q)$ such that $a_i - a_j$ is a square in $GF(q)$ if and only if $\phi(a_i) - \phi(a_j)$ is a square in $GF(q)$. In what follows we shall use the terminology and notation of Wielandt [27].

Let a be the power of a prime, say $a = p^k$. Let $\mathcal{T}(n,a)$ be the group of all permutations of $GF(a^n)$ of the form $x \to bx^\sigma$, where b is a nonzero element of $GF(a^n)$ and σ is an automorphism of $GF(a^n)$ over $GF(a)$. Let $GL(n,a)$ denote, as usual, the general linear group of all non-singular $n \times n$ matrices with entries in $GF(a)$. In a recent paper [19], D.S. Passman has proved the following result.

Theorem 1. Let $a = p^k$, where p is a prime, and suppose that G is a solvable subgroup of $GL(n,a)$ such that $\frac{a^n-1}{a^m-1} \,\Big|\, |G|$ for some divisor $m \neq n$ of n . Then either $G \leq \mathcal{T}(n,a)$ or else $(n,a) = (2,3)$, $(2,5)$, $(2,7)$, $(2,11)$, $(2,23)$, $(2,47)$, $(4,3)$ or $(6,2)$.

Let $\mathcal{S}(n,p)$ denote the group of all permutations of the elements of $GF(p^n)$ of the form $x \to sx^\sigma + b$, where s is a nonzero square of

$GF(p^n)$, σ is an automorphism of $GF(p^n)$, and b is arbitrary in $GF(p^n)$.

§3. Main Result

Theorem 2. If $q = p^n \equiv 3 \pmod 4$, then $G(R_q) = S(n,p)$.

Proof. It is clear that $S(n,p) \leq G(R_q)$. The nontrivial squares and non-squares of $GF(p^n)$ are the nodes dominated by 0 and dominating 0 , respectively. Consequently, $S_o(n,p)$, the subgroup of $S(n,p)$ fixing 0 , must permute the squares among themselves and the non-squares among themselves. Since $S_o(n,p)$ is transitive on the $\frac{q-1}{2}$ squares and on the $\frac{q-1}{2}$ non-squares, it follows that $G(R_q)$ is 3/2 transitive; hence it is either primitive or Frobenius [27; p. 25]. When $n > 1$, the automorphism group of $GF(p^n)$ is nontrivial (fixing $GF(p)$) , so that $G(R_q)$ is not Frobenius. When $n = 1$, $G(R_q)$ is also primitive because it is transitive of prime degree.

To show that $G(R_q) \leq S(n,p)$ we first consider the case $n = 1$. Then $S(1,p)$ is the group of $\binom{p}{2}$ permutations of the form $x \to sx + b$, since $GF(p)$ admits only the identity automorphism. For any $\alpha \neq \beta$ in $GF(p)$, it is well-known [27; p. 5] that

$$|G(R_q)| = |\alpha^{G(R_q)}| \cdot |\beta^{G_\alpha(R_q)}| \cdot |G_{\alpha\beta}(R_q)| \ .$$

But $|G_{\alpha\beta}| = 1$ for a solvable transitive group G of prime degree, by a result of Galois [27; p. 29]. Consequently,

$$|G(R_q)| \leq p \cdot \frac{p-1}{2} \cdot 1 = \binom{p}{2} \quad ,$$

so $G(R_q) = \mathcal{S}(1,p)$; this case may also be treated as a direct consequence of a classical theorem of Burnside (see, for example, Passman [20; p. 53]) which used the theory of group characters.

Suppose now that $n > 1$. Let A be a minimal normal subgroup of the primitive, solvable group $G = G(R_q)$. Then A is an elementary abelian p-group of order p^n [27; p. 28]. Since G is primitive, G_o is maximal. Every normal subgroup of a primitive group is transitive, so A is not contained in G_o ; hence $G = A\, G_o$. It is not difficult to show that A is its own centralizer $C(A)$ in G since A is regular and abelian. Consequently, $G_o \approx \frac{G}{C(A)}$ and this is isomorphic to a subgroup of Aut A , the automorphism group of A [25; p. 50] (Dixon [6], [7], [8] used these observations to treat other problems). Since Aut A is isomorphic to $GL(n,p)$ [25; p. 125], we may regard G_o as being a solvable subgroup of $GL(n,p)$.

Now let $m \neq n$ be any divisor of n . Clearly $\frac{p^n-1}{p^m-1}$ is an integer, since it is the index of the multiplicative group of $GF(p^m)$ in the multiplicative group of $GF(p^n)$. Since $\mathcal{S}_o(n,p) \leq G_o$ we have

$$|G_o| = t\; |\mathcal{S}_o(n,p)| = t\, n\, \frac{p^n-1}{2}$$

for some odd integer t , and it follows easily that $\frac{p^n-1}{p^m-1}$ divides $|G_o|$.

Therefore, the hypotheses of Theorem 1 are satisfied (when $k = 1$), and since n is odd we may conclude that $G_o \leq \mathcal{T}(n,p)$.

Now $G_o \neq \mathcal{T}(n,p)$ because $\mathcal{T}(n,p)$ is transitive on the nonzero elements of $GF(p^n)$. Since $\mathcal{S}_o(n,p)$ is of index 2 in $\mathcal{T}(n,p)$, it follows that $G_o = \mathcal{S}_o(n,p)$. Hence

$$|G| = |A| \cdot |G_o| = p^n \cdot n \cdot \frac{p^n-1}{2} = n\binom{p^n}{2} = |\mathcal{S}(n,p)| ,$$

and since $\mathcal{S}(n,p) \leq G$ we have that $G = G(R_q) = \mathcal{S}(n,p)$. This completes the proof of Theorem 2.

§4. An Application

Theorem 3. Let F be a finite field, where $|F| = p^n \equiv 3 \pmod 4$, and let ϕ be a permutation of F which fixes the elements of the prime field K of F; a necessary and sufficient condition that ϕ be an automorphism of F is that $\phi(a) - \phi(b)$ is a square in F if and only if $a-b$ is a square in F.

Proof. If ϕ is an automorphism of F then the condition is clearly necessary.

Theorem 2 says that the set of permutations of F satisfying the condition forms the group $G = \mathcal{S}(n,p)$. But the only elements of G which fix K are of the form $x \to x^\sigma$, where σ belongs to Aut F, since all others move either 0 or 1. This completes the proof of Theorem 3.

It can be shown that Theorem 2 is actually a special case of a result due to W.M. Kantor (to appear) which is stated without proof in the recent book by Dembowski [4 ; p. 98].

CHAPTER 4

Cycles in k-Irreducible Tournaments

§1. Introduction

A tournament T_n is k-irreducible if k is the largest integer such that for every partition of the nodes of T_n into two nonempty subsets A and B there are at least k edges that go from nodes of A to nodes of B and vice versa; a tournament T_n is irreducible if $n = 1$ or if it is k-irreducible for some positive integer k (cf. Chapter 1, §2). If a tournament T_n is not irreducible, or reducible (see Lemma 2.3, Chapter 1), then it has a unique expression of the type $T_n = A+B+\cdots+J$ where the nonempty components $A,B,\cdots,J$ are all irreducible; we call A and J the top and bottom components of T_n.

An ℓ-path (or ℓ-cycle) is a path (or cycle) of length ℓ. A spanning path or cycle of T_n is one that involves every node of T_n. An extension of a result due to Camion [3] states that if $3 \leq \ell \leq n$, then each node of any irreducible tournament T_n is contained in at least one ℓ-cycle (see [17; p. 6]). Our object in this chapter is to prove the following stronger result.

Theorem 1. Let p denote any node of any k-irreducible tournament T_n; if $3 \leq \ell \leq n$, then p is contained in at least k ℓ-cycles.

In what follows we assume that the node p and the k-irreducible tournament T_n are fixed. We may suppose that $k \geq 2$; since each node of T_n must dominate and be dominated by at least k other nodes, it follows that $2k+1 \leq n$ or $k \leq \frac{n-1}{2}$. Before proving the theorem, we make some observations about paths and the structure of the k-irreducible tournament T_n .

§2. Three Lemmas

The following result is obvious.

Lemma 2.1. Let P denote an ℓ-path from node u to node v . If node w is not contained in P and $u \to w$ and $w \to v$, then w can be inserted in the path to form an $(\ell+1)$ - path from u to v ; in particular, w can be inserted immediately before the first node of P it dominates.

Lemma 2.2. If u and v are any nodes of the top and bottom components of a reducible tournament W_t and $1 \leq \ell \leq t-1$, then there exists an ℓ-path in W_t that starts with u and ends with v ; furthermore, if $2 \leq \ell \leq t-1$ this path may be assumed to contain any given node belonging to any intermediate component of W_t .

This may be proved by applying the following observations to the components of W_t : If a tournament Z_h is irreducible and $0 \leq \ell \leq h-1$, then it contains a spanning cycle and, hence, each node is the first node, and the last node, of at least one ℓ-path in Z_h ;

and, if $R \to S$, then any c-path of R may be followed by any d-path of S to form a $(c+d+1)$ - path of $R+S$.

<u>Lemma 2.3</u>. Let G denote any minimal subtournament of the k-irreducible tournament T_n whose removal leaves a reducible tournament W of the form $W = Q+R+S$ where Q and S are irreducible and R may be empty; then each node of G is dominated by at least one node of S and dominates at least one node of Q , and there are at least k edges from nodes of G to nodes of Q and at least k edges from nodes of S to nodes of G .

The conclusions in this lemma follow from the fact that G is minimal and T_n is k-irreducible. We shall show that we may suppose such a subtournament G exists before we apply this lemma.

We now proceed to the proof of the theorem; we have to use different arguments when ℓ lies in different intervals.

§3. Proof when $\ell=3$

Let B and L denote the set of nodes that dominate p and are dominated by p , respectively. Since T_n is k-irreducible, B and L are non-empty and there are at least k edges $\overrightarrow{uv}$ that go from a node u of L to a node v of B . The theorem now follows when $\ell=3$ since each such edge $\overrightarrow{uv}$ determines a different 3-cycle $\{p,u,v,p\}$ containing p .

§4. Proof when $\ell=4$

If w is any node that dominates p, let B, L, M, and N denote the set of nodes that dominate both w and p, are dominated by both w and p, dominate w and are dominated by p, and are dominated by w and dominate p, respectively. If $L = \phi$, then M must contain at least k nodes and N must contain at least $k-1$ nodes, since p and w must each dominate at least k nodes. In this case there are at least $k(k-1) \geq k$ different 4-cycles of the type $\{p,u,w,v,p\}$ containing p, where $u \in M$ and $v \in N$. We may suppose, therefore, that $L \neq \phi$.

There are at least k edges of the type $\overrightarrow{uv}$ where $u \in L$ and $v \in B \cup M \cup N$. If $v \in B \cup M$, then the 4-cycle $\{p,u,v,w,p\}$ contains p. If $v \in N$ and v dominates some other node y of N, then the 4-cycle $\{p,u,v,y,p\}$ contains p; if there is no such node y but u is dominated by some other node z of L, then the 4-cycle $\{p,z,u,v,p\}$ contains p. Thus, there are at least k different 4-cycles containing p except, possibly, when there exists an edge $\overrightarrow{uv}$ from L to N such that u dominates the remaining nodes of L and v is dominated by the remaining nodes of N; there is at most one such edge $\overrightarrow{uv}$ so in this case the preceding construction provides at least $k-1$ 4-cycles containing p.

If $z \in M$, then $\{p,z,w,v,p\}$ is a new 4-cycle containing p. Thus we may suppose that $M = \phi$; this implies that L has at least k nodes since p dominates at least k nodes. If there exists an edge

$\overrightarrow{zy}$ where $z \neq u$, $z \in L$, and $y \in B$ then $\{p,u,z,y,p\}$ is a new 4-cycle containing p . Thus we may suppose that u is the only node of L that dominates any nodes of B . This implies, since T_n is k-irreducible, that there must be at least k edges of the type $\overrightarrow{zy}$ where $z \neq u$, $z \in L$, and $y \in N$. In this case, however, there are at least k 4-cycles of the type $\{p,u,z,y,p\}$ containing p . This completes the proof of the theorem when $\ell=4$.

§5. Proof when $5 \leq \ell \leq n-k+1$

Let C denote any $(\ell-2)$ - cycle containing p ; such a cycle exists, either by virtue of an induction hypothesis or as a consequence of the result cited in the introduction. Let B and L denote the set of nodes that dominate and are dominated by every node of C , respectively, and let M denote the set of remaining nodes of T_n that are not in C .

If $L \neq \phi$, there exist at least k edges of the type $\overrightarrow{uv}$ where $u \in L$ and $v \in B \cup M$. For each such node v there exists at least one node q of C such that $v \to q$. If we insert the nodes u and v immediately before q in C we obtain an ℓ-cycle containing p ; different edges $\overrightarrow{uv}$ clearly yield different ℓ-cycles. A similar argument may be applied to B if $B \neq \phi$, so we may now assume that $L=B=\phi$ and $M \neq \phi$.

If $u \in M$, then there exists a pair of consecutive nodes r and s of C such that $r \to u$ and $u \to s$. Thus u can be inserted

between r and s in C to form an $(\ell-1)$ - cycle C_1 containing p. Any other node v of M can now be inserted between some pair of consecutive nodes of C_1 to form an ℓ-cycle C_2 containing p. Different cycles C_2 are formed when different pairs of nodes of M are inserted in C. Thus, there are at least

$$\binom{n-(\ell-2)}{2} \geq \binom{k+1}{2} \geq k$$

different ℓ-cycles containing p when $5 \leq \ell \leq n-k+1$. (This argument can be applied for somewhat larger values of ℓ as well.)

§6. Proof when $n-k+2 \leq \ell \leq n-1$

Let T_ℓ denote any subtournament of T_n with ℓ nodes that contains the node p. If T_ℓ is irreducible, then it contains an ℓ-cycle containing p, by Camion's theorem. Thus, if each such subtournament T_ℓ is irreducible, then p is contained in at least $\binom{n-1}{\ell-1} \geq n-1 > k$ ℓ-cycles in T_n.

We may suppose, therefore, that there exists a minimal subtournament G of T_n, with $g \leq n-\ell$ nodes, whose removal leaves a reducible subtournament W containing p. Then W can be expressed in the form $W = Q+R+S$ where Q and S are irreducible and R may be empty.

There are at least k edges $\overrightarrow{xq}$ in T_n that go from a node x of G to a node q of Q, and for each such node x there exists at least one node s of S such that $s \to x$; this follows from Lemma 2.3. We shall show that for each such pair of nodes q and s, there exists an $(\ell-2)$ - path P in W that starts with q, contains the

node p , and ends with s .

If $p \in R$, then the existence of P follows immediately from Lemma 2.2 since W has $n-g$ nodes and $2 \leq \ell-2 < n-g-1$. If $p \in Q$, let P_1 denote any spanning path of Q that starts with q . We observe that if Q has m nodes then $m < \ell-2$ since otherwise the node s would be dominated by at least $\ell-2 \geq (n-k+2)-2 = n-k$ nodes and this is impossible since T_n is k-irreducible. Let P_2 denote any $(\ell-m-2)$ - path of $R+S$ that ends with s ; the existence of P_2 follows from Lemma 2.2 since $R+S$ has $n-g-m$ nodes and $1 \leq \ell-m-2 < n-g-m-1$. If $P = P_1 + P_2$, then P is an $(\ell-2)$ - path in W with the required properties and we can also find such a path when $p \in S$ by a similar argument.

This suffices to complete the proof when $n-k+2 \leq \ell \leq n-1$ since $\{x\} + P + \{x\}$ is an ℓ-cycle containing p and it is clear that different edges $\overrightarrow{xq}$ yield different ℓ-cycles.

§7. Proof when $\ell=n$

Since T_n is k-irreducible, there exists a partition of the nodes of T_n into two subsets A and B such that precisely k edges go from nodes of A to nodes of B . At least one of these subsets has more than k nodes; if the nodes in this subset that are incident with the k edges that go from A to B are removed, then the subtournament determined by the remaining nodes is reducible. It follows,

therefore, that there exists a smallest subtournament G, with at most k nodes, whose removal leaves a reducible tournament W.

Let $G = G^{(m)} + G^{(m-1)} + \cdots + G^{(2)} + G^{(1)}$ where $G^{(m)}$ and $G^{(1)}$ are irreducible and the intermediate components may be empty (in fact G itself may be irreducible), and let $W = Q + R + S$ where Q and S are irreducible and R may be empty. It follows from Lemma 2.3 that T_n contains at least k edges of the type $\overrightarrow{xy}$ where $x \in G$ and $y \in Q$.

For $x \in G^{(i)}$ we may certainly find a node $h \in G^{(m)}$ for which there exists a path $P(h,x)$ in G spanning all nodes of

$$G^{(m)} + G^{(m-1)} + \cdots + G^{(i)} ,$$

and a node $g \in S$ such that $g \to h$. In what follows let $P(y,g)$ denote a spanning path of W augmented by inserting as many nodes as possible of $G^{(i-1)} + \cdots + G^{(1)}$. Hence there exists a cycle in T_n of the form

$$C_o = \overrightarrow{xy} + P(y,g) + \overrightarrow{gh} + P(h,x) .$$

The following lemma is a direct consequence of Lemmas 2.1 and 2.3.

<u>Lemma 7.1</u>. If $w \in G^{(j)}$ cannot be inserted in $P(y,g)$, then there exist two consecutive nodes r and s of $P(y,g)$ such that w dominates all nodes of $P(y,g)$ from y to r inclusive, and is dominated

by all nodes of $P(y,g)$ from s to g inclusive.

Lemma 7.2. If $w \in G^{(j)}$, where $1 \leq j \leq i-1$, then w dominates at least two nodes of W.

Proof. Since $G^{(m)} \neq \phi$ and $|G| \leq k$, it must be that w dominates at most $k-2$ other nodes of G if $j \leq i-1$; the result now follows from the fact that w must dominate at least k nodes of T_n.

Lemmas 7.1 and 7.2 imply the following result.

Lemma 7.3. Let $i > 1$ and let A denote the set of nodes of $G^{(i-1)} + \cdots + G^{(1)}$ which cannot be inserted in $P(y,g)$; if p denotes the successor of y in $P(y,g)$, then $A \rightarrow y$, $A \rightarrow p$ and $g \rightarrow A$.

In what follows, let $P(s,z)$ denote a spanning path of the set A defined in Lemma 7.3.

Construction 1: Either $i = 1$ or $P(y,g)$ contains a node of $G^{(i-1)}$.

If $i = 1$ (this includes the possibility that G itself is irreducible) then C_o is a spanning cycle of T_n and it is obvious that distinct edges $\overrightarrow{xy}$ yield distinct spanning cycles. We may therefore suppose in what follows that $x \in G^{(i)}$ for $i > 1$.

If $i > 1$ and $P(y,g)$ contains a node of $G^{(i-1)}$, then it follows from Lemma 7.3 that the nodes of A may be inserted between x and y in C_o to yield the following spanning cycle C_1 of T_n:

$$C_1 = P(y,g) + \overrightarrow{gh} + P(h,x) + \overrightarrow{xs} + P(s,z) + \overrightarrow{zy} .$$

Beginning with a cycle C_1 of this type, we may identify x by noting that i will be the smallest index for which there exists a path P in C_1 spanning $G^{(m)} + \cdots + G^{(i)}$, and x will be the terminal node of P. Continuing along C_1 from P, we may identify y as the terminal node of the first edge encountered which goes from G to Q. It is clear that different choices of edges $\overrightarrow{xy}$, where $x \in G$, yield different spanning cycles in this way.

We henceforth suppose that $P(y,g)$ does not contain any node of $G^{(i-1)}$, and we let p and q denote the successor of y and the predecessor of g, respectively, in the path $P(y,g)$.

Construction 2: $P(h,x) \to q$.

Since $|G| \leq k$ and G is reducible, it must be that $|G^{(m)}| \leq k-1$; furthermore, it follows from the irreducibility of $G^{(m)}$ that there are at most $k-3$ nodes of G which dominate h (or zero if $k=2$). Thus, since T_n is k-irreducible, there must exist a node $t \neq y,g$ in W such that $t \to h$. But $h \to q$, so we may appeal to Lemma 2.1 to conclude that h may be inserted in $P(y,g)$ between p and q. Hence, (by Lemma 2.1 again) the successor of h in $P(h,x)$ can be inserted between h and q, and it follows that all the nodes of $P(h,x)$ can be inserted between p and q in the same order as they appeared in $P(h,x)$ originally. Denoting this augmented path by $P^*(y,g)$

we then have a spanning cycle of T_n of the type

$$C_2 = P^*(y,g) + \overrightarrow{gs} + P(s,z) + \overrightarrow{zy} \ .$$

We may identify $G^{(i-1)}$ as the component of G of highest index no node of which belongs to a sub-path of C_2 from Q to S . Thus, since C_2 maintains the order of the nodes of $P(h,x)$, we may identify x as the last node of $G^{(i)}$ encountered in a traversal of the maximal sub-path of C_2 from Q to S (y will be the initial node of this sub-path). Again, distinct choices of edges $\overrightarrow{xy}$ yield distinct spanning cycles.

<u>Construction 3</u>: There exists a node $w \in P(h,x)$ such that $q \to w$.

Let w be the first node of $P(h,x)$ which is dominated by q , let $P(w,x)$ be the sub-path of $P(h,x)$ from w to x , and let $P(p,q)$ be the sub-path of $P(y,g)$ from p to q . Consider the following cycle in T_n :

$$\overrightarrow{xy} + \overrightarrow{yg} + \overrightarrow{gs} + P(s,z) + \overrightarrow{zp} + P(p,q) + \overrightarrow{qw} + P(w,x) \quad .$$

Appealing to Lemma 2.1 and the definition of w , we may conclude that those nodes, if any, of $P(h,x)$ that precede w can all be inserted into the path $P(p,q)$ to yield a spanning cycle C_3 for T_n . There are at most two disjoint sub-paths of C_3 which start in Q and end in S and are maximal such that they do not re-enter Q after they enter S , of which $\overrightarrow{yg}$ is one.

It follows from the irreducibility of Q and S that the other path (if it exists) must contain more than one edge. Thus y may be identified as the initial node of the above-mentioned path of length one from Q to S, and x may be identified as the predecessor of y in C_3. Since $\overrightarrow{xy} \in C_3$ it is clear that different edges $\overrightarrow{xy}$ yield different spanning cycles.

To complete the proof, it will suffice to show that starting from an arbitrary spanning cycle of one of these types, it is possible to determine the construction which yielded it.

Let C be a spanning cycle obtained by one of the above methods. If C contains two disjoint paths which go from a node of Q to a node of S (whose intermediate nodes may belong to G), then C was clearly obtained by Construction 3. If this is not the case, then consider the unique maximal sub-path P in C which starts in Q and ends in S. If P contains only one edge, then since $|W| \geq 3$ it must be that C was obtained by Construction 3; hence we may suppose that P has more than one edge. If P contains a node of $G^{(m)}$, then C was obtained by Construction 2; if not, then C was obtained by Construction 1 (and it is very easy to distinguish between the possibilities for Construction 1). This completes the proof of the theorem.

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